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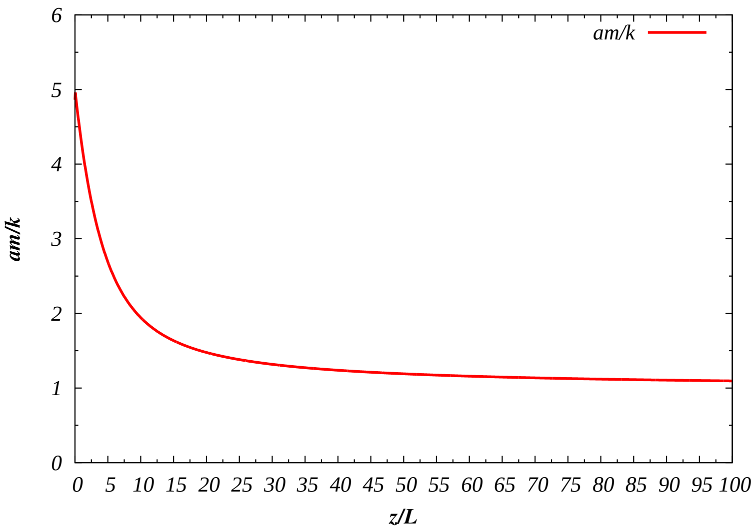

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Dansk Resume

Der foreslås en analytisk metode til beregning af turbulente overflade transporter af impuls, sensibel varme og fugtighed i det stabile grænselag. Beregningerne af de turbulente transporter benytter dimensionsløse vertikale gradienter som funktion af Monin-Obukhov (M-O) parameteren. Gradient funktionerne er empirisk bestemt ud fra feltmålinger. Beregningen af de turbulente transporter foregår i to trin. I første trin beregnes M-O stabilitetsparameteren som en entydig løsning til et tredje grads polynomium i M-O stabilitetsparameteren, som kobler denne parameter til et bulk Richardson tal og ruhedslængder for impuls og sensibel varme. I den numeriske vejrforudsigelsesmodel (NVM) beregnes bulk Richardson tallet fra model-variable. Ruhedslængderne er sædvanligvis specifiseret som inter-årligt varierende to-dimensionale felter. Tredje grads ligningen gælder som udgangspunkt for det stabilt stratifiserede horisontalt homogene og stationære grænselag og tilpasses derefter, således at dens løsninger stemmer godt overens med et statistisk baseret estimat af M-O stabilitetsparameteren som funktion af bulk Richardson tallet og ruhedsparametrene. I modsætning til den analytiske løsning har den statistiske beregning en diskontinuitet for bulk Richardson tallet lig med 0.2. I andet trin beregnes de turbulente overfladetransporter. Den hurtigste metode er først at beregne den kinematiske impulstransport, dernæst den kinematiske sensible varmetransport og til sidst den kinematiske fugtighedstransport. Beregningsmåden gør ikke brug transfer koefficienter, dette under antagelse af at de dimensionsløse profilfunktioner for sensibel varme og fugtighed er identiske.

Abstract

An analytic method, calculating turbulent surface fluxes of momentum, sensible heat and moisture in the stably stratified atmospheric boundary layer (ABL), is proposed. The calculation of the surface fluxes make use of dimensionless vertical gradients, which are universal functions of the Monin-Obukhov (M-O) stability parameter. These gradient functions are estimated empirically from field measurements. The flux calculations take place in two steps. In the first step the M-O stability parameter is obtained as a unique solution of a cubic equation, relating this parameter to a bulk Richardson number and roughness lemgths for momentum and sensible heat. In a numerical weather prediction (NWP) model the bulk Richardson number can be calculated from model variables. The roughness parameters for momentum and sensible heat are usually specified as 2-dimensional fields with inter-annual variability. The derived cubic equation is valid for the stationary, horizontal homogeneous stably stratified ABL, but is afterwards adjusted to include effects of non-stationarity (intermittent turbulence) and horizontal inhomogeneity. After the adjustment the M-O stability parameter obtained from the cubic equation is shown to become in good agreement with a statistically based estimete of the M-O stability parameter as a function of the bulk Richardson number and the roughness lengths for momentum and sensible heat. Contrary to the analytic solution, the statistical relation has a build-in discontinuity at the bulk Richardson number equal to 0.2. The turbulent surface fluxes are calculated in the second step. The fastest way is first to calculate the kinematic momentum flux, then the kinematic sensible heat flux and finally the kinematic moisture flux. Calculation of the surface fluxes does not make use of transfer coefficients if it is assumed that the non-dimensional profile functions for sensible heat and moisture are identical unique functions of the M-O stability parameter. The latter asumption is supported by the similarity between heat and moisture flux.

1 Introduction and theoretical background

Monin-Obukhov (M-O) similarity theory (Monin and Obukhov, 1954) has been widely used to describe the structure of the horizontal homogeneous and stationary surface layer. The latter is the bottom part of the atmospheric boundary layer (ABL). In the surface layer the turbulent fluxes of momentum, sensible heat and moisture can be regarded as constants, equal to their surface values. Futhermore, the impact of the Coriolis force is so small that change of wind direction with height can be ignored. In the surface layer M-O similarity theory predicts that non-dimensional vertical profiles of parameters such as mean wind speed and mean potential temperature are universal functions of the stability parameter, $\zeta = \frac{z}{L}$, where L is the Obhkhov length, defined by

$$L = \frac{(\tau_s/\rho_s)^{3/2}}{k \cdot bH_s/c_p\rho_s}. \tag{1}$$

In (1) $\tau_s = -\rho_s\overline{u'w'}$ and $H_s = -\rho_s \cdot c_p\overline{\theta' w'}$ are the turbulent surface fluxes of momentum and sensible heat, respectively, ρ_s is the air density at the surface, c_p is the specific heat capasity of air at constant pressure, $k = 0.4$ is the Von Karman constant, and finally $b \approx \frac{g}{\overline{\theta}(z)}$ is the buoyancy parameter and g is gravity. It follows from the M-O similarity hypothesis that the vertical gradients of mean wind speed and mean potential temperature can be written

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_* \phi_m}{kz} \tag{2}$$

and

$$\frac{\partial \overline{\theta}}{\partial z} = \frac{\theta_* \phi_h}{k_\theta z}, \tag{3}$$

where the friction velocity, u_* is defined as ${u_*}^2 = -\overline{u'w'}$ and the corresponding potential temperature scale, θ_*, is defined $\theta_* = -\frac{\overline{\theta' w'}}{u_*}$. The Von Karman constant k and the corresponding k_θ are defined such that $\phi_m = \phi_h = 0$ in neutral ($\zeta = 0$). M-O similarity does not provide functional forms of the ϕ-functions. The latter are determined from field measurements.

2 The bulk Richardson number

Vertical integration of the ϕ-functions in (2) and (3) from the roughness heights z_0 and $z_{0\theta}$ for wind and potential temperature, respectively, to a chosen reference height, z, within the surface layer gives

$$\overline{u}(z) = \frac{u_*}{k}(\ln(\frac{z}{z_0}) - \psi_m) \tag{4}$$

and

$$\overline{\theta}(z) - \overline{\theta}(z_{0\theta}) = \frac{\theta_*}{k_\theta}(\ln(\frac{z}{z_{0\theta}}) - \psi_h) \tag{5}$$

In (4) $\psi_m(\zeta) = \int_{\zeta_0}^{\zeta}(1-\phi_m(\zeta))dln\zeta$ and in (5) $\psi_\theta(\zeta) = \int_{\zeta_0}^{\zeta}(1-\phi_h(\zeta))dln\zeta$ with $\zeta_0 = \frac{z_0}{L}$ and $\zeta_{0\theta} = \frac{z_{0\theta}}{L}$. In numerical weather prediction (NWP) applications the reference height, z, is often chosen to be the height of the lowest model level. (4) and (5) provide the link to a bulk Richardson number, Ri_b, defined as

$$Ri_b = b \cdot \frac{(\overline{\theta}(z) - \overline{\theta}(z_{0\theta}))}{\overline{u}(z)^2} \frac{(z - z_0)^2}{z - z_{0\theta}}. \tag{6}$$

According to (4) and (5) Ri_b is a function of $\zeta, \alpha = \ln(\frac{z}{z_0})$ and $\beta = \ln(\frac{z_0}{z_{0\theta}})$. From a NWP point of view (6) has the appealing property to depend only on parameters that can be made available in a NWP model. In such a model it is necessary to calculate turbulent fluxes at the model surface. These fluxes are usually calculated by bulk transfer relations. In the early days of NWP modeling the transfer functions were specified as functions of ζ, resulting in implicit equations for the surface fluxes (e.g. Van den Hurk and Holtslag, 1997; Beljaars and Holtslag, 1991, henceforward BH1991), demanding computationally rather expensive iterations. A computationally cheaper method was proposed by Louis (1979). This method calculates the surface fluxes explicitly by specifying the transfer coefficients as functions of Ri_b and α. The latter method has been widely used and it has been refined over time in NWP modeling (Luis et al. 1982; Mascart et al., 1995; Uno et al., 1995 and Wang et al., 2002). Computation of the surface fluxes are also much simplified if ζ can be specified as a function of Ri_b and the parameters α and β. For the stably stratified ABL, Li et al. (2010), henceforward Li2010, and earlier Launiainen (1995) have proposed such a method. Li2010 base their analysis on ψ-functions proposed by BH1991, and obtain ζ as a function of Ri_b, α and β by regression analysis combined with a significance test. They do the analysis separately for the weakly stable regime ($Ri_b \leq 0.2$) and the strongly stable regime ($Ri_b > 0.2$) and the relations are shown in their equations (14) and (16) in Li2010.

3 A cubic equation analytic solution for the M-O stability parameter

In the present paper an analytic solution for ζ as function of Ri_b, α and β is proposed. The solution is based on the ϕ-functions

$$\phi_m(\zeta) = 1 + \frac{a_m}{k}\zeta \tag{7}$$

and

$$\phi_h(\zeta) = 1 + \frac{a_1\zeta + a_2\zeta^2}{1 + a_3\zeta}(1 + \frac{k}{R_\infty\zeta}). \tag{8}$$

In (7) $a_m = 2$ and in (8) $R_\infty = 0.25$, $a_1 = 0.18$, $a_2 = 0.16$ and $a_3 = 1.43$. To a good approximation (8) can be replaced by the quadratic form

$$\phi_h(\zeta) = 1 + \frac{a_{h1}}{k}\zeta + \frac{a_{h2}}{k^2}\zeta^2, \tag{9}$$

where $a_{h1} = 1.8$ and $a_{h2} = 0.18$. (7) and (8) have been proposed by Zilitinkevicch et al. (2013?). The validity of (7) to (9) is restricted to the horizontal homogeneous and stationary ABL and these functions can therefore not be expected to be valid without modifications in a stably stratified ABL influenced by horizontal inhomogeneity and non-stationarity. Such effects are at least to some extend accounted for implicitly in Li2010. However, the linear form of ϕ_m in (7) and the quadratic form of (9) have the attractive property, when substituted in (6), to give a cubic equation in ζ that can be solved analytically, giving ζ as a function of Ri_b, α and β. With a certain, not very restrictive constraint, it is found that there is only one positive solution for any Ri_b, α and β satisfying the constraint. The analytic solution is valid for the stably stratified, horizontal homogeneous and stationary ABL, and is therefore not expected to be identical with ζ obtained by Li2010.

4 The cubic equation

From (7) and (9) follow

$$-\psi_m = \frac{a_m}{k}(\zeta - \zeta_0), \tag{10}$$

and

$$-\psi_h = \frac{a_{h1}}{k}\Delta\zeta + \frac{a_{h2}}{k^2}(\zeta^2 - \zeta_{0\theta}^2), \tag{11}$$

where $\zeta_0 = \frac{z_0}{L}$, $\zeta_{0\theta} = \frac{z_{0\theta}}{L}$, $\Delta\zeta = \zeta - \zeta_{0\theta}$ and z is a reference height, which in a NWP model often is taken as the height of the lowest model level. The reference height should be chosen such that

$z >> max(z_0, z_{0\theta})$, imlpying that ψ_m and ψ_h become approximately

$$-\psi_m \approx \frac{a_m}{k}\zeta \tag{12}$$

and

$$-\psi_h \approx \frac{a_{h1}}{k}\zeta + \frac{a_{h2}}{k^2}\zeta^2. \tag{13}$$

It then follows from (2), (3), (12) and (13) that Ri_b in (6) can be written as

$$Ri_b \approx \frac{k}{k_\theta}\zeta \frac{\alpha + \beta + \frac{a_{h1}}{k}\zeta + \frac{a_{h2}}{k^2}\zeta^2}{\alpha^2 + 2\alpha\frac{a_m}{k}\zeta + (\frac{a_m}{k})^2\zeta^2}. \tag{14}$$

(14) can be rewritten as a cubic equation for ζ, given by

$$\zeta^3 + A\zeta^2 + B\zeta + C = 0, \tag{15}$$

with coefficients

$$A = \frac{k \cdot a_{h1} - k \cdot k_{0\theta}(\frac{a_m}{k})^2 Ri_b}{a_{h2}}, \tag{16}$$

$$B = \frac{k^2(\alpha + \beta) - 2k \cdot k_{0\theta}\frac{a_m}{k}\alpha Ri_b}{a_{h2}} \tag{17}$$

and

$$C = -\frac{k \cdot k_{0\theta}\alpha^2 Ri_b}{a_{h2}}. \tag{18}$$

Since $C < 0$ if $Ri_b > 0$, an investigation of the solutions to (15) shows that it is a sufficient condition for one and only one positive solution to (15) that $Ri_{bA} > Ri_{bB}$, where Ri_{bA} and Ri_{bB} are the bulk Richardson numbers at which A and B shift from positive to negative values, respectively. It follows from (16) and (17) that this condition is satisfied if $\beta < (a_{h1} - 1)\alpha$. It is noted that the combination $\frac{z}{z_0} = 100$ and $\frac{z_0}{z_{0\theta}} = 100$, which is in the parameter space considered by Li2010, does not satisfy $\beta < (a_{h1} - 1)\alpha$. In this case the fulfilment of the sufficient condition requires $z_{0\theta} > \frac{z_0}{39.81}$.

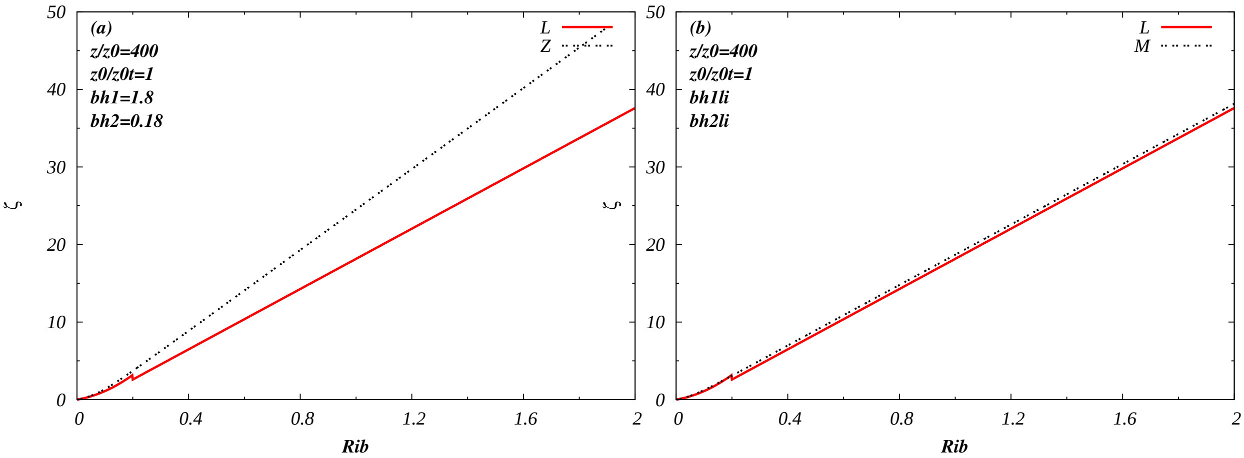

Figure 1: The M-O stability parameter, ζ, as function of the bulk Richardson number, Ri_b, for $\frac{z}{z_0} = 400$ and $\frac{z_0}{z_{0\theta}} = 1$. Curve L is the result by Li2010 and Z in (a) and M in (b) are the solutions of the cubic equation (15) without and with the modifications a_{h1m} and a_{h2m} of a_{h1} and a_{h2}, respectively.

5 Intercomparison of the M-O stability parameter obtained by the analytic and regression methods

In Figure (1), curve Z shows ζ as function of Ri_b, ranging from 0 to 2, for the case $\frac{z}{z_0} = 400$ and $\frac{z_0}{z_{0\theta}} = 1$. The curve is obtained by solving the cubic equation in (15) with the coefficients $a_m = 2$, $a_{h1} = 1.8$ and $a_{h2} = 0.18$. According to Zilitinkevich et al. (2013) curve Z, based on the profile functions (7) and (9), are proposed for the horizontal homogeneous, stationary ABL. Curve L shows the corresponding results obtained by regression analysis and a significance test by Li2010. The analysis was based on ψ_m and ψ_h functions estimated by BH1991 from field measurements in the stably stratified ABL. In the regression analysis the stable regime was divided into a weakly stable regime, $Ri_b \leq 0.2$, and a strongly stable regime $Ri_b > 0.2$. In the former regime it was assumed that ζ can be written as a sum of two terms, one depending linearly on Ri_b and the other depending quadratically on Ri_b. In the latter regime it was assumed that ζ varies linearly with Ri_b. The division by Li2010 of the stable regime into two parts introduces a discontinuity in ζ as function of Ri_b at the transition, $Ri_b = 0.2$, between the regimes, as shown in Figure (1). Intercomparison of Z and L shows an increasing spread with increasing Ri_b. Recall that curve Z represents ζ as function of Ri_b in a horizontal homogeneous, stationary ABL, while ζ represented by L is influenced by horizontal inhomogeneity and non-stationarity. Therefore, the divergence of Z and L apparently is an illustration of an increasing impact of inhomogeneity as the stability of the ABL in terms of Ri_b increases.

5.1 Implementation of inhomogeneity and nonstationarity effects in the cubic equation

In the present paper it is investigated if the solution for ζ obtained from (15) by a modification of a_{h1} and a_{h2} in (9) can be made approximately equal to ζ obtained by Li2010. If this can be done the effect of inhomogeneity and nonstationarity becomes included in the analytic solution of the cubic equation in (15) without any change in the profile function ϕ_m. The analytic solution then provides a ratio $\frac{\theta_*}{{u_*}^2}$ in fair agreement with Li2010, as shown by Figure (1b) and Figure (2).

5.2 Modification of coefficients a_{h1} and a_{h2}

The first modification, concerning a_{h2}, makes sure that the linear slopes (lapse rates) in the curves L and Z become identical. According to Li2010 the assumed linear slope in L is: $a_{s11}\alpha + a_{s21}$, with

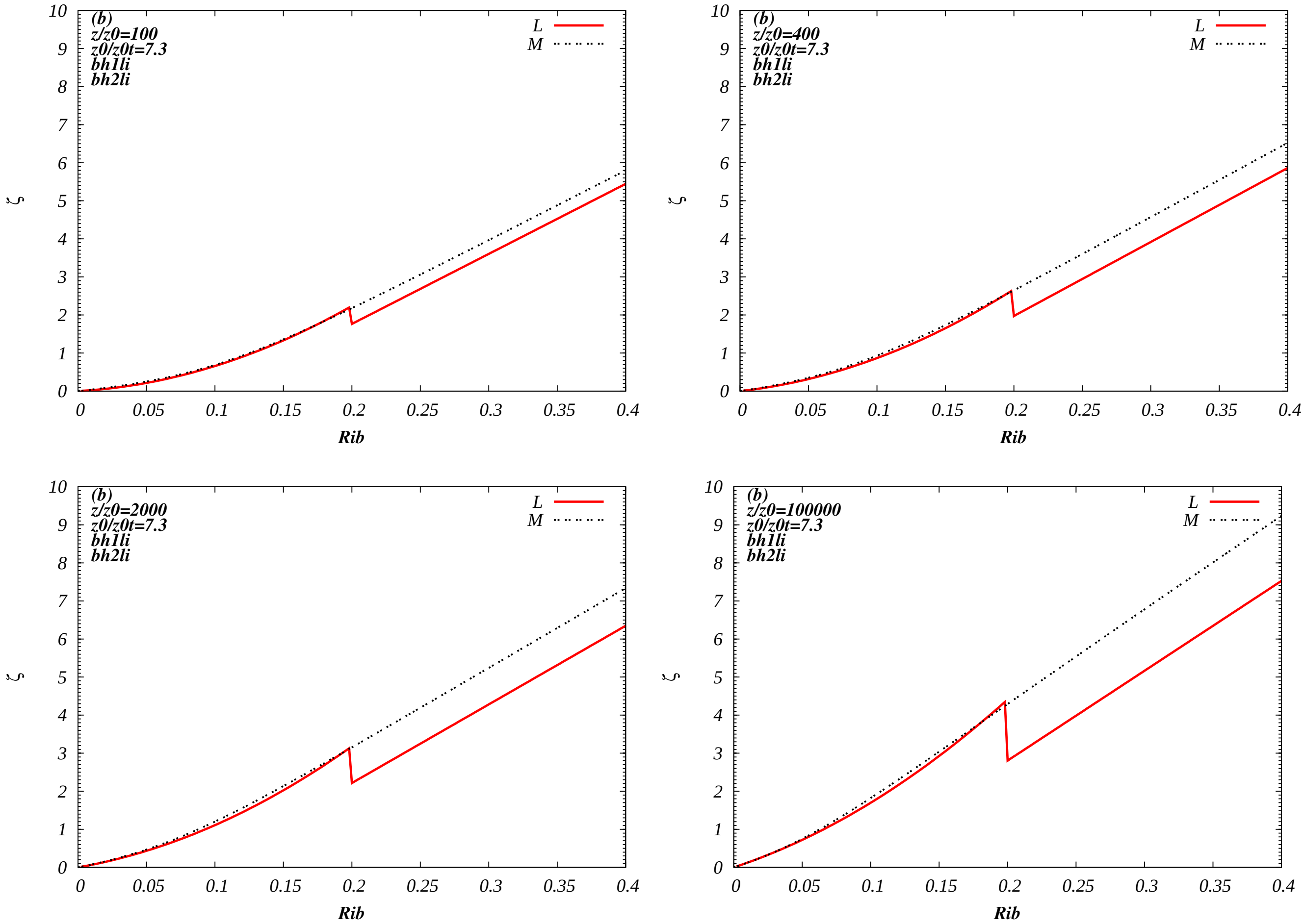

Figure 2: The M-O stability parameter, ζ, as function of the bulk Richardson number, Ri_b, for $\frac{z_0}{z_{z0\theta}} = 7.3$ and a) $\frac{z}{z_0} = 100$, b) $\frac{z}{z_0} = 400$, c) $\frac{z}{z_0} = 2000$, and d) $\frac{z}{z_0} = 100000$. Curve L hows results by Li2010 and curve M are the solutions of the cubic equation (15) with the modifications a_{h1m} and a_{h2m} of a_{h1} and a_{h2}, respectively.

$a_{s11} = 0.7529$ and $a_{s21} = 14.92$. The curve Z has the asymptotic lapse rate $\frac{k_\theta}{k}\frac{{a_m}^2}{a_{h2}}$, which follows from (14) for $\zeta \to \infty$. Identical lapse rates in L and Z therefore requires $a_{h2m} = \frac{k_\theta {a_m}^2}{k(a_{s11}\alpha + a_{s21})}$, where a_{h2m} is the modified a_{h2}. The second modification concerns a_{h1}. It was found that a_{h1} for $\frac{z_0}{z_{0\theta}} = 100$ and a_{h1} for $\frac{z_0}{z_{0\theta}} = 0.5$ in combination with a_{h2m} gave good fits to the corresponding L curves. Then, by assuming a linear relation between a_{h1} and β the modified a_{h1} reads: $a_{h1m} = a_{h1}(1.051 + 0.0734\beta)$. The solution to the cubic equation (15) with a_{h1} and a_{h2} replaced by a_{h1m} and a_{h2m}, respectively, is shown by curve M in Figure (1). It shows that M gives a fairly good fit to L in the weakly stable regime, but deviates from this curve by a nearly constant value in the strongly stable regime. The latter is due to the conflict between the discontinuity in L at Ri_b and the continuous behavior of M. Other combinations of α and β, covering the parameter space for α and β considered by Li2010, also show fairly good fits of M and L, as illustrated by Figure (2), representing rough to smooth surfaces ($\frac{z}{z_0}$ ranging from 10^2 to 10^5) and $\frac{z_0}{z_{0\theta}} = 7.3$. The sufficient condition for one and only one positive solution of (15) with the modified coefficients a_{h1m} and a_{h2m} becomes $\alpha > \frac{\beta}{0.892+0.132\beta}$, which for $\beta \geq -0.7$ is a less restrictive condition than $\alpha > \frac{\beta}{0.8}$, valid for $a_{h1} = 1.8$ and $a_{h2} = 0.18$. If, for example, $\frac{z_0}{z_{0\theta}} = 100$, $\frac{z}{z_0}$ must be larger than 316.2 wihout the modifications and larger than 21.53 with the modifications, thus allowing for a nearly 15 times larger z_0 with the modifications.

6 Calculation of turbulent surface fluxes based on the analytic solotion of the cubic equation

It has been shown in the previous section that it is possible to calculate a M-O stability parameter in fair agreement with results, based on field observations (BH1991), derived statistically as a function of Ri_b, α and β by Li2010. In the cubic equation method ζ is the unique positive solution to (15) with the coefficients a_{h1} and a_{h2} modified to a_{h1m} and a_{h2m}, respectively. Since $\frac{\theta_*}{{u_*}^2} = \frac{\zeta}{(k \cdot z \cdot b)}$, ζ only determines the ratio $\frac{\theta_*}{{u_*}^2}$. If u_* is known, θ_* is given by

$$\theta_* = \frac{{u_*}^2 \zeta}{k \cdot z \cdot b}. \tag{19}$$

If instead θ_* is known, u_* is given by

$${u_*}^2 = \frac{\theta_* k \cdot z \cdot b}{\zeta}. \tag{20}$$

The computation of the turbulent kinematic momentum and sensible heat flux, u_*^2 and $u_*\theta_*$, respectively, is therefore done in two steps. In the first step u_* is obtained from

$$u_* = \frac{k\overline{V}(z)}{\alpha - \psi_{mBH}} \tag{21}$$

and in the second step θ_* can be calculated from (19). Alternatively, θ_* can be calculated in the first step from

$$\theta_* = \frac{k_\theta(\overline{\theta}(z) - \overline{\theta}(z_{0\theta}))}{\alpha + \beta - \psi_{hBH}}, \tag{22}$$

and then u_* in the second step from (20). In (21) and (22) ψ_{mBH} and ψ_{hBH} are the ψ-functions for momentum and sensible heat, respectively, estimated by BH1991 and applied in the regression analysis by Li2010. According to BH1991

$$-\psi_{mBH} = a\zeta + b(\zeta - \frac{c}{d})exp(-d\zeta) + \frac{bc}{d}, \tag{23}$$

and

$$-\psi_{hBH} = (1 + \frac{2}{3}a\zeta)^{3/2} + b(\zeta - \frac{c}{d})exp(-d\zeta) + \frac{bc}{d} - 1, \tag{24}$$

with $a = 1$, $b = 0.667$, $c = 5$ and $d = 0.35$. Calculation of θ_* in the first step requires knowledge of $\overline{\theta}(z_{0\theta})$.

Note that if u_* has been calculated from (21) and θ_* afterwards from (19), an approximation to $\overline{\theta}(z_{0\theta})$ can be obtained from

$$\overline{\theta}(z_{0\theta}) = \overline{\theta}(z) - \frac{\theta_*}{k}(\alpha + \beta - \psi_{hBH}). \tag{25}$$

The advantages of calculating u_* in the first step instead of θ_* are slightly less calculations, but first of all no need for specification of $\overline{\theta}(z_{0\theta})$. Finally, the kinematic turbulent surface moisture flux, q_*u_*, can be calculated from the bulk transfer relations $\theta_*u_* = C_H\overline{V}(\overline{\theta}(z) - \overline{\theta_s})$ and $q_*u_* = C_Q\overline{V}(\overline{q}(z) - \overline{q_s})$, giving

$$q_*u_* = \frac{C_Q(\overline{q}(z) - \overline{q}_s)}{C_H(\overline{\theta}(z) - \overline{\theta}_s)}\theta_* u_*. \tag{26}$$

In (26) C_H and C_Q are transfer coefficients for sensible heat and moisture, respectively, and it has been assumed that $\overline{\theta}(z_{0\theta})$ and $\overline{q}(z_{0q})$ can be approximated by the surface values $\overline{\theta_s}$ and $\overline{q_s}$, respectively.

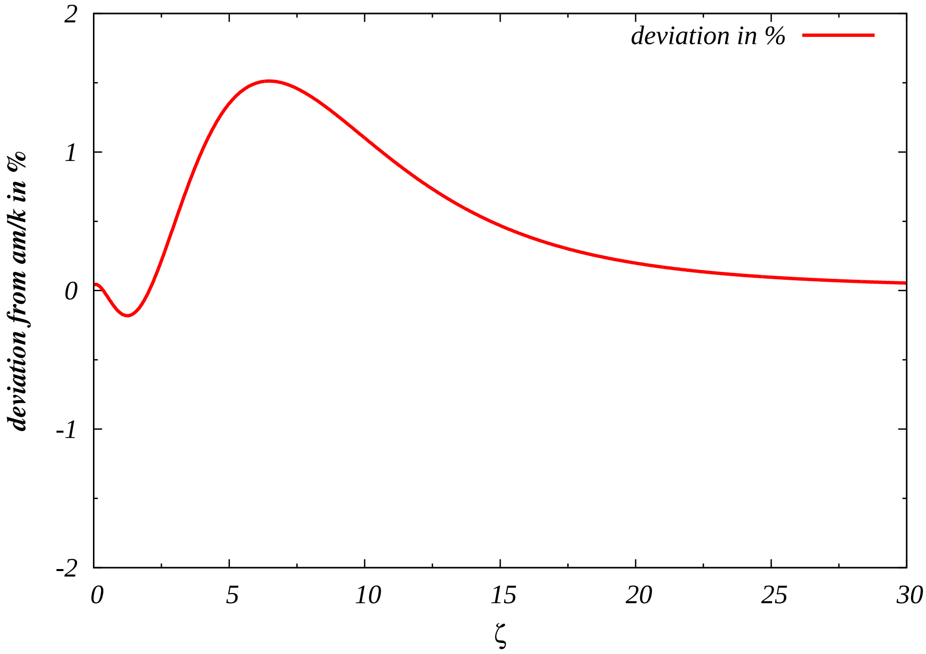

Figure 3: Deviation in % of the approximation (27) from $-\psi_{mBH} = \frac{a_m(\zeta)}{k}\zeta$.

To obtain the kinematic moisture flux the ratio $\frac{C_Q}{C_H}$ must be known. Field measurements and similaity between water vapor and heat transfer suggest $C_H = C_Q$ (e.g. Arya, 2002), which is equivalent to assuming $\phi_h = \phi_q$ and $z_{0q} = z_{0\theta}$.

Finally, it is noted that formally $-\psi_{mBH}$ can be written as $-\psi_{mBH} = \frac{a_m(\zeta)}{k}\zeta$. The front-page figure shows the variation of $\frac{a_m(\zeta)}{k}$ with ζ. It follows from (23) that for small values of ζ the approximation $\frac{a_m(\zeta)}{k} \approx a + b(1 + c - d \cdot \zeta)$ can be used, and for large values of ζ a good approximation is $\frac{a_m(\zeta)}{k} \approx a + \frac{bc}{d}\zeta^{-1}$, confirming that $\frac{a_m(\zeta)}{k}$ decreases from 5 for $\zeta = 0$ and approaches 1 asymptotically for $\zeta \to \infty$. If appropriate, an approximation to $\frac{a_m(\zeta)}{k}$, like

$$\frac{a_m(\zeta)}{k} \approx a + \frac{c_b}{\zeta + f(\zeta)}, \tag{27}$$

with $c_a = \frac{c}{d(c+1)}$, $c_b = \frac{bc}{d}$ and $f(\zeta) = \frac{c_a}{1+(0.09+0.018\zeta)c_a \cdot \zeta}$, can be applied. Figure (3) shows that the approximation above deviates less than about 1.5% from $\frac{a_m(\zeta)}{k}$.

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