THE FREQUENCY DISTRIBUTION OF THE DATES OF EASTER

Michael E. Davison, Computer Centre, The Queen's University of Belfast

Abstract

From the rules for the determination of Easter it is not easy to answer questions such as, "Is an early or late Easter as likely as one in the middle of the range?" or, "Does one date occur more frequently than any other?" These rules are explained and from them the frequency distribution of the dates is derived. This shows that an early or late Easter is not as likely as one in the middle of the range and that one date (April 19) does occur more frequently than any other.

1. *Introduction*

From earliest times, Christians have held an annual celebration of the day on which their Lord rose from the dead. Unfortunately, from an almost equally early time there was a difference of opinion concerning when this celebration should be held. Some chose a Sunday at the time of the Jewish Passover; others, the Quartodecimans, celebrated the 14th of the Jewish month Nisan irrespective of the day of the week on which the 14th fell. Further, it was felt that Easter should depend on the moon since the Jewish months were lunar dependent, but by the beginning of the fourth century A.D. various methods were in use for calculating the phases of the moon. All of this led to much confusion.

Following the Council of Nicaea in 325 A.D. a rule was drawn up which did not, however, meet with general acceptance until about the 8th century A.D. This rule defined Easter as "The Sunday following the full moon which falls on or next after the vernal equinox". This rule must be qualified as follows:

(1) For the purpose of determining Easter, the phases of the moon were assumed to have a cycle time of 19 years, i.e. if the moon was full at a certain time on a certain date, it would be full at the same time on the same date 19 years later.

(2) The moon was assumed to be full on the 14th day.

(3) The date of the vernal equinox was taken to be 21st March.

One consequence of this rule was that the limits of Easter were fixed as the 22nd March and 25th April both inclusive.

Under the Julian Calendar, which was then in force, a leap year occurred every fourth year (which is no longer the case! See below). This property of the Julian Calendar together with the 19-year Lunar Cycle made the determination of Easter relatively (!) straightforward and resulted in the dates of Easter repeating themselves every $4 \times 7 \times 19 = 532$ years.

2. *The Gregorian Reformation of the Calendar*

In the centuries following the Council of Nicaea, the true equinox began to fall before the 21st March and the actual full moons began to fall after the date indicated by the 19 year Lunar Cycle. Attention began to be given to this problem as early as the 13th century but it was not until the Gregorian Reformation of the Calendar, which came into effect in Italy in 1582, that steps were taken to correct the errors which had occurred and to prevent a similar accumulation of errors in the future.

By the 16th century, the actual equinox was falling on the 11th March (10 days before the assumed equinox) and to return the equinox to the 21st March, 10 days were dropped from the calendar. Also the actual full moons were falling four days after the calculated full moons and a similar adjustment was made in the Lunar Calendar to correct this error. We now examine in more detail how these errors were prevented from recurring.

2.1 *The Solar Correction*

The true length of the Solar year is about 365 days, 5 hours, 49 minutes, i.e. about 11 minutes less than the Julian year of 365 days, 6 hours. This discrepancy is corrected in the Gregorian Calendar by the suppression of three intercalary (leap) days in 400 years. This is known as the *Solar Correction* and is effected by the rule that an end-of-century year must be a multiple of 400 to be a leap year. Thus 1900 was not a leap year but 2000 will be.

2.2 *The Lunar Correction*

The true cycle time of the moon's phases is less (by about 1 hour 28 minutes) than 19 Julian years, the error amounting to about 8 days in 2,500 years. The correction is implemented by altering the dates of the full moons every 300 years to make them fall one day earlier. This correction is applied seven times consecutively and an eighth time after 400 years. Thus the total correction of eight days in 2,500 years is effected. This is known as the *Lunar Correction.*

3. *Easter in the Gregorian Calendar*

Having seen how corrections were made to the Solar and Lunar cycles, we now describe how Easter is calculated under the (current) Gregorian Calendar. To find Easter for any year we need to know (1) the dates on which the Sundays fall during the year, and (2) the dates on which the full moons fall during the year.

3.1 *The Sunday Letter*

The dates on which the Sundays fall in any year are given by the *Sunday Letter*. The meaning of this is as follows. The days of the year were numbered (1 to 365 for a common year) and then the seven letters A to G were attached repeatedly to these days as shown:

1	2	3	4	5	6	7	8	9	…	…
A	B	C	D	E	F	G	A	B	…	…

If the first Sunday of the year falls on January 1st, then A is the Sunday Letter of the year and every day which appears against an A is a Sunday. Similarly if the first Sunday falls on January 2nd then B is the Sunday Letter of the year and so on.

If the year in question is a leap year it has two Sunday letters—one applying to January and February, the other applying to the rest of the year.

Butcher (1877, p. 139) gives the following formula for the Sunday letter.

$$L = [(7 + X_1 - \sigma_1)/7]_r \quad (1)$$

where $X_1 = [\{X + (X/4)_e\}/7]_r$, $\sigma_1 = [\{\sigma - (\sigma/4)_e\}/7]_r$, X is the year, σ the centurial year (19 for this century), $_r$ denotes remainder to be taken and $_e$ denotes whole part to be taken. Having found L, the Sunday letter is found from the following table

L	0	1	2	3	4	5	6
Sunday Letter	*A*	*G*	*F*	*E*	*D*	*C*	*B*

Thus for 1980 we have $X_1 = [(1980 + 495)/7]_r = 4$, and $\sigma_1 = [(19-4)/7]_r = 1$. Therefore, $L = [(7 + 4 - 1)/7]_r = 3$, and Sunday Letter $= E$. When the year in question is a leap year (as 1980) the Sunday letter found by this method is that which applies to March to December. The letter applying to January and February 1980 is *F* indicating that the first Sunday in January fell on the sixth which is indeed the case.

The Sunday letter for 1981 is *D* and here we see the origin of the term "leap" year—the Sunday letter has *leapt* two places from its value at the beginning of 1980, the letters appearing in retrograde order.

3.2 *The Epact*

Starting from an arbitrary year and knowing the dates of the full moons in that year, we can work out the dates of all subsequent full moons in the 19-year cycle taking the time of a lunation to be 29 and 30 days alternately. The true time of a lunation is about $29\frac{1}{2}$ days. The position in the 19-year Lunar Cycle which any given year occupies is known as the *Golden Number* of the year. Thus knowing the Golden Number of the year we know when the full moons will fall during that year.

The full moons are also indicated by the *Epact* which is the age of the moon (in days) at the beginning of the year. Thus a year with N (Golden Number) = III might have Epact 21; the following year, $N =$ IV, would have Epact 2 etc. The Epacts in any century have a fixed relation to the Golden Numbers as follows:

N	III	IV	V	VI
Epact	21	2	13	24

i.e. *Epact* $= [(11\,N - 12)/30]_r$ where N is the Golden Number and $_r$ indicates remainder to be taken as before. Thus the Epact could be substituted for the Golden Number in the calculation of Easter.

However, when an intercalary day is suppressed (three times in 400 years) we must diminish the Epacts by one for the following century since the moon is one day younger than it would otherwise have been. This produces another Epact series as follows.

N	III	IV	V	VI
Epact	20	1	12	23

When the Lunar Correction is made (approx. once in 300 years) the Epacts are increased by one since the actual full moon is falling about one day before the calculated full moon.

It can be seen that repeated application of the Solar and Lunar Corrections will produce several Epact series. In general, each series is valid for 100 years until a further correction forces us to use a new series. The total number of Epact series is 30 since the age of the moon can vary only between 0 and 29 days, an age of 30 days being taken as equivalent to 0.

Table 1.

THE EXPANDED TABLE OF EPACTS.

Index to Centuries. After Correction.					Index Letters.	Golden Numbers.																		
						I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.
17	18	87	88	89	C	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18
	19	20	21	90	B	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17
22	24	91	92	93	A	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16
	23	25	94	96	u	27	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15
26	27	28	95	97	t	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14
29	30	98	99	100	s	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13
31	32	33	102		r	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	20	1	12
34	36	103	104	105	q	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11
35	37				p	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10
38	39	40			n	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9
41					m	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8
42	43	44			l	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7
45	46				k	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6
47	48	49			i	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5
50	52				h	16	27	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4
51	53				g	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3
54	55	56			f	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2
57	58				e	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	20	1
59	60	61			d	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*
62	64				c	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29
63	65				b	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28
66	68				a	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27
67	69				P	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26
70	71	72			N	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22	3	14	25′
73	74				M	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24
75	76	77			H	5	16	27	8	19	*	11	22	3	14	25	6	17	28	9	20	1	12	23
78	80				G	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	*	11	22
79	81				F	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21
82	83	84			E	2	13	24	5	16	27	8	19	*	11	22	3	14	25′	6	17	28	9	20
1582	16	85	86		D	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19

These 30 series together with the centuries to which they apply are shown in Table 1 (reproduced from Butcher's book). The asterisk indicates an age of 0 or 30 days, taken as equivalent. (The occurrence of Epact 25′ is connected with the desire not to have a full moon falling twice on the same day in the same 19-year cycle). Hence we see that series *B* is in use this century.

Now the formula for the Golden Number is $N = [(X + 1)/19]_r$, giving $N = 5$ for 1980. From series *B*, the Epact corresponding to $N = \mathrm{V}$ is 13 and so this is the Epact for 1980. We now refer to Table 2 (also reproduced from Butcher's book) and find that Sunday letter *E* and Epact 13 give 6th April for Easter Day.

Table 2.

THE GREGORIAN PASCHAL TABLE

Showing the days on which Easter can fall according to the possible combinations of the Epacts and Sunday Letters.

Dom. Lett.	Cycle of Epacts.	Easter Day.
D	23	22 M.
	22 21 20 19 18 17 16	29 M.
	15 14 13 12 11 10 9	5 A.
	8 7 6 5 4 3 2	12 A.
	1 * 29 28 27 26 25′ 25 24	19 A.
E	23 22	23 M.
	21 20 19 18 17 16 15	30 M.
	14 13 12 11 10 9 8	6 A.
	7 6 5 4 3 2 1	13 A.
	* 29 28 27 26 25′ 25 24	20 A.
F	23 22 21	24 M.
	20 19 18 17 16 15 14	31 M.
	13 12 11 10 9 8 7	7 A.
	6 5 4 3 2 1 *	14 A.
	29 28 27 26 25′ 25 24	21 A.
G	23 22 21 20	25 M.
	19 18 17 16 15 14 13	1 A.
	12 11 10 9 8 7 6	8 A.
	5 4 3 2 1 * 29	15 A.
	28 27 26 25′ 25 24	22 A.
A	23 22 21 20 19	26 M.
	18 17 16 15 14 13 12	2 A.
	11 10 9 8 7 6 5	9 A.
	4 3 2 1 * 29 28	16 A.
	27 26 25′ 25 24	23 A.
B	23 22 21 20 19 18	27 M.
	17 16 15 14 13 12 11	3 A.
	10 9 8 7 6 5 4	10 A.
	3 2 1 * 29 28 27	17 A.
	26 25′ 25 24	24 A.
C	23 22 21 20 19 18 17	28 M.
	16 15 14 13 12 11 10	4 A.
	9 8 7 6 5 4 3	11 A.
	2 1 * 29 28 27 26 25′	18 A.
	25 24	25 A.

NOTE—Whenever the Epact is 25′, which can only be when the Golden Number > 11, we *must* change 25′ to 26.

Whenever the Epact is 24 we *may* change it into 25; but with Letter C this change is *necessary*.

4. *The Frequency Distribution of the Dates of Easter*

4.1 *The Distribution of the Sunday Letter*

Referring again to the formula for the Sunday letter, equation (1), it may be shown (by adding 400 to X) that this letter will cycle in 400 years. This fact may be given a physical explanation as follows:

Since a common year is 52 weeks and one day the number of "extra" days in 400 years is

400 + 100 (intercalary days) — 3 (intercalary days suppressed) = 497.

But 497 days is exactly 71 weeks, therefore 400 years contains an integral number of weeks and so the Sundays will fall on the same dates after this period, as they did 400 years earlier.

Using a computer program, the author has shown for a specific 400-year period (1700—2099) that the frequencies of the Sunday Letters are the following

A	*G*	*F*	*E*	*D*	*C*	*B*
56	58	57	57	58	56	58

but since 400 years is the cycle time this distribution will be the same for any 400 year period. If we were to move our range to 1701—2100 we would lose one *C* (the Sunday letter for 1700) and gain one *C* (Sunday letter for 2100); therefore the distribution would be constant.

4.2. *The Distribution of the Epacts*

As already stated, the Solar Correction is applied to the Epacts three times in 400 years and the Lunar Correction eight times in 2,500 years. Thus in 10,000 years the Solar Correction would cause the Epact to be reduced by 75; the Lunar Correction would cause them to be increased by 32 giving a net reduction of 43. Reducing the Epact by 43 days is equivalent to a full cycle through the 30 series of Epacts together with a reduction of 13 days Therefore, it would require 30 periods of 10,000 years (i.e. 300,000 years) to make an integral number of cycles through the 30 Epact series.

For the Solar Correction, Butcher (p. 130) gives the formula $C_s = [3(\sigma—15)/4]_e$, and for the Lunar Correction (p. 133) $C_l = [\sigma—15—a)/3]_e$, where $a = [(\sigma—17)/25]_e$ and σ is the centurial year as before. For some centuries, only the Solar Correction will be applied, for some only the Lunar, for some both (cancelling each other) and for some neither.

Again using a computer program, the author has shown for a specific period of 3,000 centuries (centuries 17 to 3016) that each of the Epact series occurs an equal number of times during the period. By reasoning in a similar way to the case of the distribution of the Sunday Letter we see that this equal distribution of the Epact series will hold true for any 300,000 year period.

4.3 *The Distribution of Easter Dates*

We have seen that after 300,000 years we shall have returned to the same Epact series as at the beginning of the period, but as the Epact for a particular year depends also on the Golden Number it will take 19 × 300,000 = 5,700,000 years

Table 3.

Frequency Distribution of Easter for Cycle Time of 5700000 years.

Date of Easter		Frequency of Occurrence	Percentage Probability	Frequency in terms of p, q, r *
22 March	...	27550	0.483	*p*
23	...	54150	0.950	2*q*
24	...	81225	1.425	3*q*
25	...	110200	1.933	4*p*
26	...	133000	2.333	5*r*
27	...	165300	2.900	6*p*
28	...	186200	3.267	7*r*
29	...	192850	3.383	7*p*
30	...	189525	3.325	7*q*
31	...	189525	3.325	7*q*
1 April	...	192850	3.383	7*p*
2	...	186200	3.267	7*r*
3	...	192850	3.383	7*p*
4	...	186200	3.267	7*r*
5	...	192850	3.383	7*p*
6	...	189525	3.325	7*q*
7	...	189525	3.325	7*q*
8	...	192850	3.383	7*p*
9	...	186200	3.267	7*r*
10	...	192850	3.383	7*p*
11	...	186200	3.267	7*r*
12	...	192850	3.383	7*p*
13	...	189525	3.325	7*q*
14	...	189525	3.325	7*q*
15	...	192850	3.383	7*p*
16	...	186200	3.267	7*r*
17	...	192850	3.383	7*p*
18	...	197400	3.463	7 8/19 *r*
19	...	220400	3.867	8*p*
20	...	189525	3.325	7*q*
21	...	162450	2.850	6*q*
22	...	137750	2.417	5*p*
23	...	106400	1.867	4*r*
24	...	82650	1.450	3*p*
25	...	42000	0.737	1 11/19 *r*

* *p*, *q*, and *r* are equal to 27,550, 27,075, and 26,600 respectively. These are the frequencies which would occur if a particular date were produced by only one Epact, and by a Sunday Letter probability of 58/400, 57/400, and 56/400 respectively.

before we return to the same Epact series and the same Golden Number. As 5,700,000 is a multiple of 400 (the cycle time of the Sunday Letter) we see that the cycle time of Easter itself is 5,700,000 years. This cycle time is quoted in the Explanatory Supplement to the Astronomical Ephemeris.

Referring to Table 2 which gives the dates of Easter produced by all combinations of Epacts and Sunday Letters, we see that only Sunday Letter *D* and Epact 23 will produce 22nd March for Easter. The probability of Sunday Letter *D* occurring is (from 4.1) 58/400. Now the probability of an Epact series being in use which contains 23 is 19/30 since (Table 1) 19 of the 30 series contain 23 and each series is equally likely to occur. The probability of a Golden Number being in use which will produce Epact 23 is 1/19. Therefore the expected number of occurrences of 22nd March during the Easter cycle is: $58/400 \times 19/30 \times 1/19 \times 5{,}700{,}000 = 27{,}550$.

Similarly we see that e.g. 29th March, 5th April and 12th April are seven times more likely to occur since they are produced by seven Epacts and will thus occur 192,850 times.

Proceeding in a similar fashion we produce column two of Table 3, noticing that Epact 25 occurs only 11 times and 25′ only 8 times. Column 3 gives the percentage probability of occurrence of the different dates and column 4 gives the frequencies in terms of the frequencies which would occur if a particular date were produced by only one Epact and by a Sunday Letter probability of 58/400, 57/400, and 56/400 respectively. Figure 1 gives the same information in the form of a histogram. The frequencies in Table 3 were also produced by computer by evaluating the perpetually valid formula (Table 4) for 5,700,000 years!*

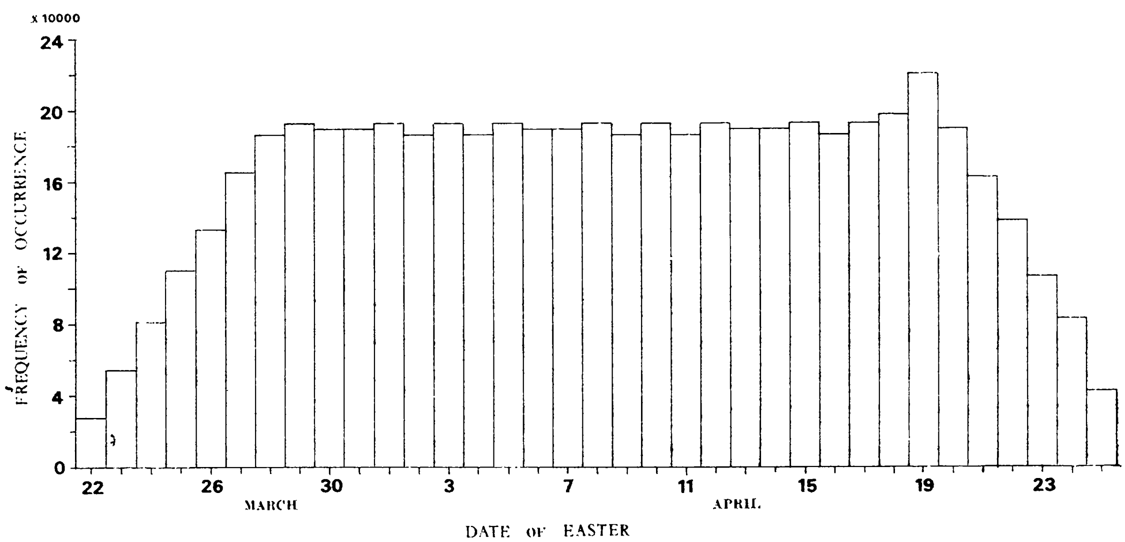

Figure 1. Distribution of the dates of Easter over a period of 5,700,000 years.

* This formula appeared in the edition of "*Nature*" dated 20 April 1876 and was reproduced in Butcher's book.

Table 4.

Determining Easter for any year (n is the number of the month of the year and $p + 1$ is the number of the day of the month on which Easter falls).

Divide		By	And call the Quotient	Remainder
The year X	...	19	—	a
X	...	100	b	c
b	...	4	d	e
$b + 8$	...	25	f	—
$b - f + 1$	...	3	g	—
$19a + b - d - g + 15$	...	30	—	h
c	...	4	i	k
$32 + 2e + 2i - h - k$	...	7	—	l
$a + 11h + 22l$	...	451	m	—
$h + l - 7m + 114$	...	31	n	p

It is clear from Table 3 that some dates appear much more frequently than others and in particular the most frequently occurring date (19th April) occurs eight times more often than the least frequently occurring date (22nd March). Easter will not again fall on 22nd March until 2285.

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Mr. Michael E. Davison, Computer Centre, The Queen's University of Belfast, Belfast BT7 1NN, Northern Ireland.